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Monochromatic 4-term arithmetic progressions in 2-colorings of $\mathbb{Z}_n$ ☆

Linyuan Lu, Xing Peng

University of South Carolina, Columbia, SC 29208, United States

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A B S T R A C T

This paper is motivated by a recent result of Wolf on the minimum number of monochromatic 4-term arithmetic progressions (4-APs, for short) in $\mathbb{Z}_p$, where p is a prime number. Wolf proved that there is a 2-coloring of $\mathbb{Z}_p$ with 0.000386% fewer monochromatic 4-APs than random 2-colorings; the proof is probabilistic. In this paper, we present an explicit and simple construction of a 2-coloring with 9.3% fewer monochromatic 4-APs than random 2-colorings. This problem leads us to consider the minimum number of monochromatic 4-APs in $\mathbb{Z}_n$ for general n. We obtain both lower bound and upper bound on the minimum number of monochromatic 4-APs in $\mathbb{Z}_n$. Wolf proved that any 2-coloring of $\mathbb{Z}_p$ has at least $(1/16 + o(1))p^2$ monochromatic 4-APs. We improve this lower bound to $(7/96 + o(1))p^2$.

Our method for $\mathbb{Z}_n$ naturally apply to the similar problem on $[n]$. In 2008, Parillo, Robertson, and Saracino constructed a 2-coloring of $[n]$ with 14.6% fewer monochromatic 3-APs than random 2-colorings. In 2010, Butler, Costello, and Graham used a new method to construct a 2-coloring of $[n]$ with 17.35% fewer monochromatic 4-APs (and 26.8% fewer monochromatic 5-APs) than random 2-colorings. Our construction gives a 2-coloring of $[n]$ with 33.33% fewer monochromatic 4-APs (and 57.89% fewer monochromatic 5-APs) than random 2-colorings.

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E-mail addresses: lu@math.sc.edu (L. Lu), pengx@mailbox.sc.edu (X. Peng).

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1. Introduction

Let G be a finite subset of a commutative group. For any integer $k \geqslant 3$, a *k-term arithmetic progression* (or k-AP, for short) is an (ordered) sequence of k elements in G of the form $(a, a+d, \ldots, a+(k-1)d)$, where a is the *first element* and d is the *common difference*. A 2-coloring of G is a map $c: G \to \{0, 1\}$. A k-AP $(a, a+d, \ldots, a+(k-1)d)$ is *monochromatic* if $c(a) = c(a+d) = \cdots = c(a+(k-1)d)$. Let $m_k(G, c)$ be the number of monochromatic k-APs in the 2-coloring c. A natural question is: how small can $m_k(G, c)$ be? Let $\mathrm{AP}_k(G)$ be the number of all k-APs in G. Define

$$m_k(G) := \min_c \frac{m_k(G, c)}{\mathrm{AP}_k(G)}. \tag{1}$$

We are interested in the asymptotic value of $m_k(G)$ as $|G|$ approaches infinity. (This is similar to questions on Schur triples [3,7,9] or on general patterns [1,4].)

In this paper, we consider only the cases $G = [n]$ and $G = \mathbb{Z}_n$. Here $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{Z}_n$ is the cyclic group of order n. When n is a prime number p, we write $\mathbb{Z}_n$ as $\mathbb{Z}_p$ for emphasis. A k-AP is *degenerate* if it contains repeated terms; it is *non-degenerate* otherwise. The *mirror image* of a k-AP $(a, a+d, \ldots, a+(k-1)d)$ is another k-AP $(a+(k-1)d, \ldots, a+d, a)$. Here we allow k-APs to be degenerate; a k-AP will be considered different from its mirror image except for $d = 0$. In contrast, many papers require k-APs to be non-degenerate and treat a k-AP the same as its mirror image. The two different definitions of k-APs derive two different versions of $m_k(G)$. However, they are asymptotically equivalent as $|G|$ goes to infinity; this is because the number of degenerate k-APs is only $O(n)$ while the number of all APs is $\Omega(n^2)$. A k-AP $(a, a+d, \ldots, a+(k-1)d)$ is parametrized by a pair (a, d). The parameter space of all k-APs in $[n]$ can be described as $\{(a, d)\colon\ 1 \leqslant a \leqslant n,\ 1 \leqslant a+(k-1)d \leqslant n\}$. A k-AP $(a, a+d, \ldots, a+(k-1)d)$ in $[n]$ is degenerate if and only if $d = 0$. The parameter space of all k-APs in $\mathbb{Z}_n$ is simply $\mathbb{Z}_n^2$. A k-AP $(a, a+d, \ldots, a+(k-1)d)$ in $\mathbb{Z}_n$ is degenerate if $jd \equiv 0 \mod n$ for some $0 \leqslant j \leqslant k-1$. In both cases, the number of degenerate k-APs is $O(n)$.

Random 2-colorings of $[n]$ (or $\mathbb{Z}_n$) give the following upper bounds:

$$m_k\big([n]\big) \leqslant 2^{1-k} + o(1); \tag{2}$$

$$m_k(\mathbb{Z}_n) \leqslant 2^{1-k} + o(1). \tag{3}$$

Van der Waerden's number [10] $W = W(2, k)$ is the smallest positive integer n such that any two coloring of $[n]$ has at least one monochromatic k-AP. We can use W to show a lower bound on $m_k([n])$. For example, using a double counting method, we can prove $m_k([n]) \geqslant \frac{2(k-1)}{W^3} + o(1)$ (see [1]). By a similar argument we can show $m_k(\mathbb{Z}_n) \geqslant \frac{2(k-1)}{W^2} + o(1)$. These bounds are usually too weak; stronger bounds exist for $k = 3$ and $k = 4$.

The case $\mathbb{Z}_p$ is of particular interest. The number of monochromatic 3-APs in $\mathbb{Z}_p$ depends only on the size of the coloring classes, but not on the coloring itself (see [3,8]). Namely, if c is a 2-coloring of $\mathbb{Z}_p$ such that the size of red classes is αp, then we have

$$m_3(\mathbb{Z}_p, c) = \big(1 - 3\alpha + 3\alpha^2\big)p^2. \tag{4}$$

The minimum is attained at $\alpha = \frac{1}{2}$. Thus $m_3(\mathbb{Z}_p)$ is achieved by random 2-colorings.

For $k = 4$, Wolf [11] proved that for any sufficiently large prime number p, we have

$$\frac{1}{16} + o(1) \leqslant m_4(\mathbb{Z}_p) \leqslant \frac{1}{8}\left(1 - \frac{1}{259\,200}\right) + o(1). \tag{5}$$

This lower bound improved a previous lower bound due to Cameron, Cilleruelo, and Serra [2], who proved

$$m_4(\mathbb{Z}_n) \geqslant \frac{2}{33} + o(1), \tag{6}$$

where n is relatively prime to 6 and large enough. (Cameron, Cilleruelo, and Serra's result actually holds for any Abelian group of order n provided $\gcd(n, 6) = 1$.)

Wolf's upper bound indicates that $m_4(\mathbb{Z}_p)$ is not achieved by random 2-colorings. However, the quantity is only slightly less than $\frac{1}{8}$ — the density of monochromatic 4-APs in random 2-colorings. Her method for the upper bound relies heavily on the method introduced by Gowers (see [11]). The existence of such 2-coloring is proved by probabilistic methods; i.e., it is non-constructive.

To get a better upper bound for $m_k(\mathbb{Z}_n)$, we introduce a construction consisting of periodic blocks. For a fixed b, let B be a 2-coloring of $\mathbb{Z}_b$ with $m_k(\mathbb{Z}_b)b^2$ monochromatic k-APs. (Here B is viewed as a 0-1 string of length b.)

Write $n = bt + r$ with $0 \leqslant r \leqslant b - 1$. We consider the following periodic construction c

$$\underbrace{BB\cdots B}_{t}R. \tag{7}$$

Here R is any bit-string of length r.

If n is divisible by b, then R is empty. In this case, it is easy to see that the periodic construction above gives $m_k(\mathbb{Z}_b)n^2$ monochromatic k-APs. Thus, we have

$$m_k(\mathbb{Z}_n) \leqslant m_k(\mathbb{Z}_b) \quad \text{if } b|n. \tag{8}$$

If n is not divisible by b, then the computation of $m_k(\mathbb{Z}_n, c)$ is more complicated in general. Note that the number of k-APs containing some element(s) in R is a lower order term as n goes to infinity; the value $m_k(\mathbb{Z}_n, c)$ can be still determined asymptotically by B and t. (See Lemmas 3.2 and 3.3.)

We say two colorings c and c' of $\mathbb{Z}_n$ are *isomorphic* if there is an integer m such that $\gcd(m, n) = 1$ and $c'(v) = c(mv)$ for any $v \in \mathbb{Z}_n$. Two colorings c and c' of $\mathbb{Z}_n$ are *conjugate* if $c'(v) = 1 - c(v)$ for any $v \in \mathbb{Z}_n$. It is clear that $m_k(\mathbb{Z}_n, c) = m_k(\mathbb{Z}_n, c')$, whenever c and c' are isomorphic or conjugate to each other. To find the coloring B, we implement an efficient breadth-first search algorithm to reduce isomorphic copies. Using this efficient program, we found a 2-coloring B_{20} of $\mathbb{Z}_{20}$ for 4-APs,

$$B_{20} = (1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0).$$

This coloring B_{20} gives $m_4(\mathbb{Z}_{20}) = \frac{9}{100}$.

We also searched for a coloring of $\mathbb{Z}_p$, with p small, containing no non-degenerate monochromatic 4-APs. At $p = 11$, there is a unique coloring with this property up to isomorphism. Since 0's and 1's are not balanced in this coloring, we considered the 2-coloring B_{22} of $\mathbb{Z}_{22}$ instead, where

$$B_{22} = (1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0).$$

In this coloring, all monochromatic 4-APs are degenerate; there are 22 monochromatic 4-APs with $d = 0$ and 20 monochromatic 4-APs with $d = 11$. This coloring B_{22} gives $m_4(\mathbb{Z}_{22}) = \frac{42}{22^2} = \frac{21}{242}$.

The following theorem improves both Wolf's lower bound and upper bound on $m_4(\mathbb{Z}_p)$. Our lower bound is obtained by combining Wolf's elegant method and an exhaustive search. Our upper bound is proved by a novel method of analyzing the number of monochromatic k-APs in the periodic construction (7).

Theorem 1.1. *If p is prime and large enough, then we have*

$$0.07291666 < \frac{7}{96} \leqslant m_4(\mathbb{Z}_p) \leqslant \frac{17}{150} + o(1) < 0.1133334. \tag{9}$$

In fact, our methods naturally lead to asymptotic bounds on $m_4(\mathbb{Z}_n)$ for general n. The results depend on n by case. For simplicity, we split it into two theorems; one on the lower bound and the other one on the upper bound.

Theorem 1.2. *If n is sufficiently large, then we have*

$$m_4(\mathbb{Z}_n) \geqslant \begin{cases} \frac{7}{96} & \text{if } n \text{ is not divisible by } 4, \\ \frac{2}{33} & \text{if } n \text{ is divisible by } 4. \end{cases}$$

Here is the theorem for the upper bound on $m_4(\mathbb{Z}_n)$ for general n.

Theorem 1.3. *If n is sufficiently large, then we have*

$$m_4(\mathbb{Z}_n) \leqslant \begin{cases} \frac{17}{150} + o(1) < 0.1133334 & \text{if } n \text{ is odd}, \\ \frac{8543}{72\,600} + o(1) < 0.1176722 & \text{if } n \text{ is even}. \end{cases}$$

Note that Theorem 1.1 is a corollary of Theorems 1.2 and 1.3. The upper bound above is small enough to beat the bound $\frac{1}{8}$ given by random 2-colorings. Using inequality (8), we can get a much better bound for certain values of n. For example,

$$m_4(\mathbb{Z}_n) \leqslant \begin{cases} 0.09 & \text{if } 20|n, \\ 0.086777 & \text{if } 22|n. \end{cases}$$

For $m_5(\mathbb{Z}_n)$, we use the periodic construction with the following coloring of $\mathbb{Z}_{74}$:

$$\begin{aligned} B_{74} = (&1,1,1,1,0,1,1,1,0,0,0,0,1,0,1,1,0,0,1,0,1,0,1,0,0,1,1,0,1,0,0,0,0,\\ &1,1,1,0,1,1,1,1,0,1,1,1,0,0,0,0,1,0,1,1,0,0,1,0,0,0,1,0,0,1,1,0,1,\\ &0,0,0,0,1,1,1,0). \end{aligned}$$

All monochromatic 5-APs in B_{74} are degenerate ones; there are 74 of them with $d = 0$ and 72 of them with $d = 37$. This coloring gives $m_5(\mathbb{Z}_{74}) = \frac{146}{74^2} = \frac{73}{2738}$. We have the following theorem.

Theorem 1.4. *If n is sufficiently large, then we have*

$$m_5(\mathbb{Z}_n) \leqslant \begin{cases} \frac{3629}{65\,712} + o(1) < 0.055226 & \text{if } n \text{ is odd}, \\ \frac{3647}{65\,712} + o(1) < 0.0554998 & \text{if } n \text{ is even}. \end{cases}$$

Once again, the upper bound above is small enough to beat the bound $\frac{1}{16}$ given by random 2-colorings. Using inequality (8), we can get a much better upper bound for certain values of n. For example, we have

$$m_5(\mathbb{Z}_n) \leqslant \frac{73}{2738} = 0.026661\ldots \quad \text{if } 74|n.$$

We have the following theorem.

Theorem 1.5. *We have*

$$\varliminf_{n\to\infty} m_4(\mathbb{Z}_n) \leqslant \frac{1}{12}, \qquad \varliminf_{n\to\infty} m_5(\mathbb{Z}_n) \leqslant \frac{1}{38}.$$

In fact, we show for any ε, there is an odd integer n with $m_4(\mathbb{Z}_n) \leqslant \frac{1}{12} + \varepsilon$. Combining this result with Theorem 1.2, we get

$$\frac{7}{96} \leqslant \inf\{m_4(\mathbb{Z}_n)\colon n \text{ is not divisible by } 4\} \leqslant \frac{1}{12}.$$

Note the gap between the upper bound and the lower bound is small. Here we conjecture that the upper bound is tight.

Conjecture 1.1. $\inf\{m_4(\mathbb{Z}_n)\} = \frac{1}{12}$.

The periodic construction also works for $m_k([n])$. We have

Lemma 1.1. *For any $k \geqslant 3$ and any positive integer b, we have*

$$\overline{\lim_{n\to\infty}} m_k([n]) \leqslant m_k(\mathbb{Z}_b).$$

In particular, we have

$$\overline{\lim_{n\to\infty}} m_k([n]) \leqslant \underline{\lim}_{n\to\infty} m_k(\mathbb{Z}_n).$$

When we consider similar problems for $[n]$, the k-AP and its mirror image are often not distinguished in the literature. To avoid the ambiguity, we call a k-AP (in $[n]$) with $d > 0$ an *increasing* k-AP. For $k \geqslant 3$, let c_k be the largest number satisfying "for any $\varepsilon > 0$, there is a sufficiently large n such that any 2-coloring of $[n]$ contains at least $(c_k - \varepsilon)n^2$ monochromatic increasing k-APs". Since $[n]$ has $(\frac{1}{2(k-1)} + o(1))n^2$ increasing k-APs, it is equivalent to say

$$c_k = \frac{1}{2(k-1)} \overline{\lim_{n\to\infty}} m_k([n]). \tag{10}$$

In 2008, Parillo, Robertson, and Saracino [5] proved

$$0.05111 < \frac{1675}{32\,768} \leqslant c_3 \leqslant \frac{117}{2192} < 0.053376. \tag{11}$$

Recently, Butler, Costello, and Graham [1] proved $c_4 < 0.0172202\ldots$ and $c_5 < 0.005719619\ldots$. Both bounds beat random 2-colorings.

Combining Theorem 1.5 with Lemma 1.1, we have

$$c_4 \leqslant \frac{1}{72} = 0.01388888\ldots, \tag{12}$$

$$c_5 \leqslant \frac{1}{304} = 0.003289474\ldots. \tag{13}$$

These numerical results indicate that the periodic construction is often better than the block construction used in [1]. We believe the following conjecture holds.

Conjecture 1.2. *For fixed $k \geqslant 4$, we have $\overline{\lim}_{n\to\infty} m_k([n]) = \underline{\lim}_{n\to\infty} m_k(\mathbb{Z}_n)$.*

Bounding c_3 is very different from bounding c_4. The conjecture above is not true for $k = 3$. We have the following property.

Property 1. *If the integer n is large enough, then any 2-coloring of $\mathbb{Z}_n$ contains at least $\frac{1}{4}n^2$ monochromatic 3-term arithmetic progressions, i.e.,*

$$m_3(\mathbb{Z}_n) = \frac{1}{4} + o(1). \tag{14}$$

With the help of computer search, we found three 2-colorings B_{20}, B_{22}, and B_{74}, which are used as building blocks in constructing 2-colorings of $\mathbb{Z}_n$ and $[n]$. The data in Tables 1, 2, 3, and 4 can be easily verified by anyone with limited programming experience. Lower bounds on $m_4(\mathbb{Z}_n)$ and $m_5(\mathbb{Z}_n)$ require non-trivial exhaustive search in the same way as Cameron, Cilleruelo, and Serra [2] proving the inequality (6). However the exhaustive computer search is not the focus of this paper.

The organization of the paper is the following. In Section 2, we will prove a necessary lemma and Theorem 1. In Section 3, we first prove a lemma and a corollary on counting lattice points in a polygon; then we prove Theorem 1.3 for odd n and Theorem 1.4. In Section 4, we introduce a recursive construction and then use it to prove Theorem 1.5 and Theorem 1.3 for even n. In the last section, we will deal with the lower bounds and prove Theorem 1.2.

2. Notation and the proof of Property 1

Let $c : \mathbb{Z}_n \to \{0,1\}$ be a 2-coloring of $\mathbb{Z}_n$. The coloring c is often viewed as a bit-string of length n. For convenience, we say an element $v \in \mathbb{Z}_n$ is *red* if $c(v) = 0$ and *blue* if $c(v) = 1$. The coloring c induces a partition $\mathbb{Z}_n = A \cup B$, where A is the set of red elements while B is the set of blue elements.

Let $k \geqslant 3$ be an integer and $|A| = \alpha n$. We have $|B| = (1-\alpha)n$.

For each $1 \leqslant i \leqslant k$, let A_i (or B_i) be the set of all k-APs whose i-th term is red (or blue), respectively; we have

$$|A_i| = \alpha n^2, \tag{15}$$

$$|B_i| = (1-\alpha)n^2. \tag{16}$$

Lemma 2.1. *For $1 \leqslant i < j \leqslant k$, if $\gcd(j-i, n) = 1$, then we have*

$$|A_i \cap A_j| = \alpha^2 n^2, \tag{17}$$

$$|B_i \cap B_j| = (1-\alpha)^2 n^2. \tag{18}$$

If $\gcd(j-i, n) \neq 1$, then we have

$$|A_i \cap A_j| \geqslant \alpha^2 n^2, \tag{19}$$

$$|B_i \cap B_j| \geqslant (1-\alpha)^2 n^2. \tag{20}$$

Proof. For $1 \leqslant i < j \leqslant k$, the value of $|A_i \cap A_j|$ equals the number of k-APs whose i-th and j-th terms are red. If $\gcd(j-i, n) = 1$, then every ordered pair of elements (distinct or not) in $\mathbb{Z}_n$ can be extended into a unique k-AP whose i-th and j-th terms are the given pair. Note that the number of ordered pairs of red (and blue) elements is exactly $\alpha^2 n^2$ (and $(1-\alpha)^2 n^2$), respectively. Eqs. (17) and (18) follow.

If $\gcd(j-i, n) \neq 1$, then every pair of elements in $\mathbb{Z}_n$ may or may not be extended into a k-AP whose i-th and j-th terms are the given pair. Let $r = \gcd(j-i, n)$. For $0 \leqslant l \leqslant r-1$, let x_l be the number of elements z in $\mathbb{Z}_n$ such that z is red and $z \equiv l \mod r$. For any pair (z_1, z_2), the elements z_1 and z_2 are the i-th and j-th elements of an arithmetic progression if

$$z_2 - z_1 = (j-i)d, \tag{21}$$

for some element d in $\mathbb{Z}_n$. Equivalently, $z_2 - z_1 \equiv 0 \mod r$. Moreover, if $z_2 - z_1 \equiv 0 \mod r$, then Eq. (21) has r solutions. It follows that

$$|A_i \cap A_j| = r \sum_{l=0}^{r-1} x_l^2 \geqslant \left(\sum_{l=0}^{r-1} x_l \right)^2 = \alpha^2 n^2. \tag{22}$$

Eq. (20) can be proved similarly. $\square$

Proof of Property 1. Observe that if we assign red and blue to each number equally likely, then the expected value of $m_3(\mathbb{Z}_n, c)$ is $n^2/4 + O(n)$. Therefore, there is a 2-coloring c such that $m_3(\mathbb{Z}_n, c) \leqslant n^2/4 + O(n)$, that is $m_3(\mathbb{Z}_n) \leqslant 1/4 + O(1/n)$.

For the other direction, let c be any 2-coloring of $\mathbb{Z}_n$. We use the notation α, A_i, and B_i defined in the beginning of this section.

We have the following inclusion–exclusion formula:

$$|A_1 \cup A_2 \cup A_3| = \sum_{i=1}^{3} |A_i| - \sum_{1 \leqslant i < j \leqslant 3} |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|. \tag{23}$$

Note that $A_1 \cup A_2 \cup A_3 = \overline{B_1 \cap B_2 \cap B_3}$ and $|\overline{B_1 \cap B_2 \cap B_3}| = n^2 - |B_1 \cap B_2 \cap B_3|$. By the definition of $m_3(\mathbb{Z}_n, c)$, we have $|A_1 \cap A_2 \cap A_3| + |B_1 \cap B_2 \cap B_3| = m_3(\mathbb{Z}_n, c)$. Applying Lemma 2.1, we have

$$m_3(\mathbb{Z}_n, c) = n^2 - \sum_{i=1}^{3} |A_i| + \sum_{1 \leqslant i < j \leqslant 3} |A_i \cap A_j| \geqslant n^2 - 3\alpha n^2 + 3\alpha^2 n^2 = \bigl(1 - 3\alpha(1-\alpha)\bigr)n^2.$$

Note that $\alpha(1-\alpha)$ reaches the maximum value at $\alpha = 1/2$. We have $m_3(\mathbb{Z}_n, c) \geqslant n^2/4$. Therefore $m_3(\mathbb{Z}_n) \geqslant 1/4$ and the lemma follows. $\square$

3. Proofs of Theorem 1.3 for odd *n* and Theorem 1.4

In this section, we will examine the number of monochromatic k-APs in the periodic construction (7).

3.1. Proof of Lemma 1.1

We need a tool to count the grid points inside a polygon on the plane.

A point in $\mathbb{R}^2$ is a *grid point* if both coordinates are integers. Let Q be a simple polygon whose vertices are grid points. Let $A(Q)$ be the area of Q, $I(Q)$ be the number of grid points inside Q, and $B(Q)$ be the number of grid points on the boundary of Q. Pick's theorem [6] states

$$A(Q) = I(Q) + \frac{B(Q)}{2} - 1.$$

Intuitively, if $B(Q)$ is a lower order term, then $I(Q) \approx A(Q)$. Let P be a simple polygon in the plane $\mathbb{R}^2$. For any $t > 0$ and a point v, a new polygon $v + tP$ is obtained by first scaling P by a factor of t and then translating it by a vector v. We have the following lemma, whose proof is elementary and is given in Appendix A.

Lemma 3.1. *Suppose P is a simple polygon with m vertices and circumference L. For any vector v and sufficiently large t, we have*

$$\bigl|I(v + tP) - A(P)t^2\bigr| = O(t).$$

By counting grid points in $(-x_0/b, -y_0/b) + (n/b)P$, we get the following corollary. Since the number of grid points on the boundary of nP is always a lower order term, it does not matter whether grid points on the boundary are included or not.

Corollary 3.1. *For any fixed point (x_0, y_0), let L_b be a lattice $\{(x_0 + ib, y_0 + jb)\colon i, j \in \mathbb{Z}\}$ and P be a simple polygon. For $n \gg b$, we have*

$$|L_b \cap nP| = \frac{n^2}{b^2} A(P) + O\left(\frac{n}{b}\right).$$

Proof of Lemma 1.1. Let B be a 2-coloring/bit-string of $\mathbb{Z}_b$ with $m_k(\mathbb{Z}_b)b^2$ monochromatic k-APs. Any k-AP in $\mathbb{Z}_b$ can be parametrized by a pair (a', d') satisfying $0 \leqslant a', d' \leqslant b - 1$. Let S be the set of parameters (a', d') such that the corresponding k-APs in $\mathbb{Z}_b$ are monochromatic. We have

$$|S| = m_k(\mathbb{Z}_b)b^2.$$

For sufficiently large n, we write $n = bt + r$ with $0 \leqslant r \leqslant b - 1$. Consider the periodic construction $BB \cdots BR$ (see (7)). Note that the number of k-APs containing some elements of R is $O(n)$. We need to estimate the number of monochromatic k-APs lying entirely in $[bt]$.

Let P be a parallelogram defined by

$$P = \bigl\{(x, y)\colon 0 < x \leqslant 1 \text{ and } 0 < x + y(k-1) \leqslant 1\bigr\}.$$

The area of P is clearly $\frac{1}{(k-1)}$.

A k-AP $(a, a+d, \ldots, a+(k-1)d)$ in $[bt]$ is monochromatic if and only if $(a \bmod b,\ d \bmod b) \in S$. Let $L_b^{(a',d')}$ be the lattice $\{(a'+ib, d'+jb):\ i, j \in \mathbb{Z}\}$. Applying Corollary 3.1, the number of monochromatic k-APs in $[bt]$ is exactly

$$\sum_{(a',d')\in S} \left|L_b^{(a',d')} \cap (bt)P\right| = \sum_{(a',d')\in S} A(P)t^2 + O(t) = |S|A(P)t^2 + O\big(b^2 t\big)$$
$$= \frac{1}{k-1} m_k(\mathbb{Z}_b)(bt)^2 + O\big(b^2 t\big).$$

Thus,

$$m_k\big([n], c\big) = \frac{1}{k-1} m_k(\mathbb{Z}_b)(bt)^2 + O\big(b^2 t\big).$$

Note that the number of k-APs in $[n]$ is $\frac{n^2}{k-1} + O(n)$. Taking the ratio, we have

$$m_k\big([n]\big) \leqslant \frac{m_k([n], c)}{\mathrm{AP}([n])} = m_k(\mathbb{Z}_b) + O\left(\frac{1}{t}\right).$$

First taking (upper) limit as n goes to infinity, we get

$$\varlimsup_{n\to\infty} m_k\big([n]\big) \leqslant m_k(\mathbb{Z}_b).$$

Then taking (lower) limit as b goes to infinity, we have

$$\varlimsup_{n\to\infty} m_k\big([n]\big) \leqslant \varliminf_{b\to\infty} m_k(\mathbb{Z}_b).$$

The proof of Lemma 1.1 is finished. $\square$

3.2. *Proof of Theorem 1.3 for odd n*

It suffices to consider the case that n is not divisible by b. Write $n = bt + r$ with $1 \leqslant r \leqslant b-1$. Recall the periodic construction

$$\underbrace{BB\cdots B}_{t} R.$$

Here R is any bit-string of length r.

The number of 4-APs containing some bit(s) in R is $O(n)$; we need only to consider the monochromatic 4-APs containing no bit from R. We divide the set of all non-degenerate 4-APs in $\mathbb{Z}_n$ into eight classes C_i for $0 \leqslant i \leqslant 7$. Note that a 4-AP is determined by the pair (a, d) satisfying $0 \leqslant a, d \leqslant n-1$. These eight classes are defined as follows:

Classes	Meanings
C_0	$a < a+d < a+2d < a+3d < n$
C_1	$a < a+d < a+2d < n \leqslant a+3d < 2n$
C_2	$a < a+d < n \leqslant a+2d < a+3d < 2n$
C_3	$a < a+d < n \leqslant a+2d < 2n \leqslant a+3d < 3n$
C_4	$a < n \leqslant a+d < a+2d < a+3d < 2n$
C_5	$a < n \leqslant a+d < a+2d < 2n \leqslant a+3d < 3n$
C_6	$a < n \leqslant a+d < 2n \leqslant a+2d < a+3d < 3n$
C_7	$a < n \leqslant a+d < 2n \leqslant a+2d < 3n \leqslant a+3d < 4n$

These 8 classes can be viewed as 8 regions in the parameter space of (a, d) as shown in Fig. 1. Let us normalize the parameters so that the area of the whole square is 1. For $0 \leqslant i \leqslant 7$, let a_i be the area of the i-th normalized region. We have

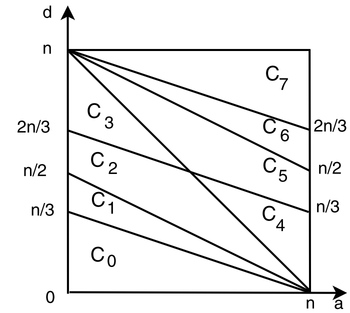

Fig. 1. The eight regions in the parameter space of all 4-APs in $\mathbb{Z}_n$.

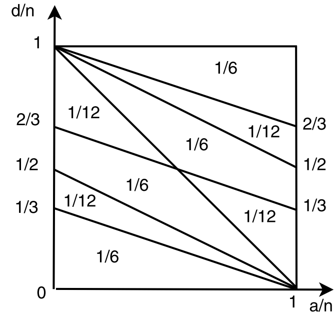

Fig. 2. The areas of the eight normalized regions.

$$a_0 = \frac{1}{6}, \qquad a_1 = \frac{1}{12}, \qquad a_2 = \frac{1}{6}, \qquad a_3 = \frac{1}{12},$$
$$a_4 = \frac{1}{12}, \qquad a_5 = \frac{1}{6}, \qquad a_6 = \frac{1}{12}, \qquad a_7 = \frac{1}{6}.$$

For $r_1, r_2, r_3 \geqslant 0$, an (r_1, r_2, r_3)-*generalized* 4-*term arithmetic progression* is of form

$$a, \qquad a + d - r_1, \qquad a + 2d - (r_1 + r_2), \qquad a + 3d - (r_1 + r_2 + r_3).$$

Here (a, d) are the parameters determining the (r_1, r_2, r_3)-generalized 4-term arithmetic progression.

We have the following lemma:

Lemma 3.2. *For $0 \leqslant i \leqslant 7$, write i as a bit-string $x_1x_2x_3$ of length three. Let c_i be the number of all monochromatic (x_1r, x_2r, x_3r)-generalized 4-term arithmetic progressions in B. Then the number of monochromatic 4-APs in $BB \cdots BR$ is*

$$\sum_{i=0}^{7} a_i c_i t^2 + O(t).$$

In particular, we have

$$m_4(\mathbb{Z}_n) \leqslant \sum_{i=0}^{7} a_i \frac{c_i}{b^2} + o(1).$$

Proof. A 4-AP is said to be on *the boundary of some* C_i if it is in C_i and contains an element in R. Note that the number of 4-APs on the boundary is $O(n)$. We can ignore these 4-APs in the calculation below.

For any $0 \leqslant a', d' \leqslant b - 1$, The lattice $L_b^{(a',d')} = \{(a' + ub, d' + vb)\colon\ 0 \leqslant u, v < t\}$ distributes evenly in the square $[0, n) \times [0, n)$. Applying Corollary 3.1, we have

$$\left|L_b^{(a',d')} \cap C_i\right| = a_i t^2 + O(t)$$

for $0 \leqslant i \leqslant 7$. We also observe any monochromatic 4-term arithmetic progression with parameter $(a, d) = (a' + ub, d' + vb) \in C_i$ if and only if the (x_1r, x_2r, x_3r)-generalized 4-term arithmetic progression with parameter (a', d') is monochromatic in B. Thus the number of monochromatic 4-term arithmetic progressions with parameter $(a, d) \in C_i$ is

$$c_i a_i t^2 + O(t).$$

Hence the number of monochromatic 4-term arithmetic progressions in $BB \cdots BR$ is $\sum_{i=0}^{7} a_i c_i t^2 + O(t)$ and $m_4(\mathbb{Z}_n) \leqslant \sum_{i=0}^{7} a_i c_i / b_i^2 + O(1/n)$. $\square$

Table 1
The values of c_i's for B_{20} and any odd r satisfying $1 \leqslant r \leqslant 19$.

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
36	50	50	50	50	50	50	36

Table 2
The values of c_i's for B_{22} and each even r such that $2 \leqslant r \leqslant 20$.

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
42	63	70	63	63	70	63	42

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Recall $B_{20} = (1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0)$ and $B_{22} = (1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0)$.

If n is odd, we use the periodic construction $B_{20}B_{20}\cdots B_{20}R$. We write $n = 20t + r$, where $r = 1, 3, 5, \dots, 19$. For each odd r, it turns out that the value of c_i depends only on i, not on r. These values are given in Table 1.

By Lemma 3.2, we have

$$m_4(\mathbb{Z}_n) \leqslant \sum_{i=0}^{7} \frac{a_i c_i}{b^2} + o(1) = \frac{17}{150} + o(1).$$

If n is even, we prove only a weaker result $m_4(\mathbb{Z}_n) \leqslant \frac{175}{1452} + o(1) < 0.12052342$ here and *postpone the proof of the stated bound in the theorem until the end of next section*. We write $n = 22t + r$ where $r = 0, 2, 4, \dots, 20$. We use the periodic construction $B_{22}B_{22}\cdots B_{22}R$. If $r = 0$, then we have

$$m_4(\mathbb{Z}_n) \leqslant m_4(\mathbb{Z}_{22}) = \frac{21}{242} < 0.086777.$$

We are through in this case. For $r = 2, 4, 6, \dots, 20$, the value c_i depends only on i, not on r. These values are given in Table 2.

By Lemma 3.2, we have

$$m_4(\mathbb{Z}_n) \leqslant \sum_{i=0}^{7} \frac{a_i c_i}{b^2} + o(1) = \frac{175}{1452} + o(1). \qquad \square$$

3.3. *Proof of Theorem 1.4*

In this section, we will consider the 5-APs. Let B be a 2-coloring of $\mathbb{Z}_b$. We consider the periodic construction $c = BB\cdots BR$.

Similar to the proof of Theorem 1.3, we can divide all non-degenerate 5-APs into 14 classes C_i with index i in $S = \{0, \dots, 15\} \setminus \{3, 12\}$, see table below and Fig. 3; let a_i be the area of i-th normalized region (see Fig. 4). We have

$$\begin{aligned}
&a_0 = \frac{1}{8}, \quad a_1 = \frac{1}{24}, \quad a_2 = \frac{1}{12}, \quad a_4 = \frac{1}{12}, \quad a_5 = \frac{1}{12}, \\
&a_6 = \frac{1}{24}, \quad a_7 = \frac{1}{24}, \quad a_8 = \frac{1}{24}, \quad a_9 = \frac{1}{24}, \quad a_{10} = \frac{1}{12}, \\
&a_{11} = \frac{1}{12}, \quad a_{13} = \frac{1}{12}, \quad a_{14} = \frac{1}{24}, \quad a_{15} = \frac{1}{8}.
\end{aligned}$$

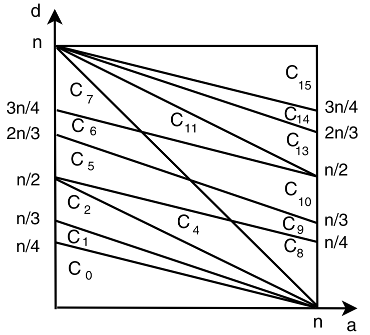

Fig. 3. The 14 regions of the parameter space of all 5-APs in $\mathbb{Z}_n$.

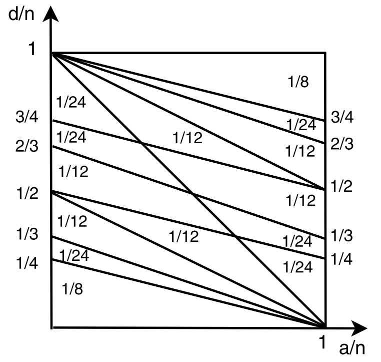

Fig. 4. The areas of the 14 normalized regions.

Classes	Meanings
C_0	$a < a+d < a+2d < a+3d < a+4d < n$
C_1	$a < a+d < a+2d < a+3d < n \leqslant a+4d < 2n$
C_2	$a < a+d < a+2d < n \leqslant a+3d < a+4d < 2n$
C_4	$a < a+d < n \leqslant a+2d < a+3d < a+4d < 2n$
C_5	$a < a+d < n \leqslant a+2d < a+3d < 2n \leqslant a+4d < 3n$
C_6	$a < a+d < n \leqslant a+2d < 2n \leqslant a+3d < a+4d < 3n$
C_7	$a < a+d < n \leqslant a+2d < 2n \leqslant a+3d < 3n \leqslant a+4d < 4n$
C_8	$a < n \leqslant a+d < a+2d < a+3d < a+4d < 2n$
C_9	$a < n \leqslant a+d < a+2d < a+3d < 2n \leqslant a+4d < 3n$
C_{10}	$a < n \leqslant a+d < a+2d < 2n \leqslant a+3d < a+4d < 3n$
C_{11}	$a < n \leqslant a+d < a+2d < 2n \leqslant a+3d < 3n \leqslant a+4d < 4n$
C_{13}	$a < n \leqslant a+d < 2n \leqslant a+2d < a+3d < 3n \leqslant a+4d < 4n$
C_{14}	$a < n \leqslant a+d < 2n \leqslant a+2d < 3n \leqslant a+3d < a+4d < 4n$
C_{15}	$a < n \leqslant a+d < 2n \leqslant a+2d < 3n \leqslant a+3d < 4n \leqslant a+4d < 5n$

Assume $r_i \geqslant 0$ for $1 \leqslant i \leqslant 4$. An (r_1, r_2, r_3, r_4)-*generalized* 5-*term arithmetic progression* is of form

$$a, \qquad a+d-r_1, \qquad a+2d-(r_1+r_2), \qquad a+3d-\sum_{i=1}^{3} r_i, \qquad a+4d-\sum_{i=1}^{4} r_i.$$

Given (r_1, r_2, r_3, r_4), an (r_1, r_2, r_3, r_4)-generalized 5-term arithmetic progression is determined by (a, d). We have the following lemma; we will omit the proof since it is similar to Lemma 3.2.

Lemma 3.3. *Let $S = \{0, \ldots, 15\} \setminus \{3, 12\}$. For each $i \in S$, write i as a bit-string $x_1x_2x_3x_4$ of length four. Let c_i be the number of all monochromatic (x_1r, x_2r, x_3r, x_4r)-generalized 5-term arithmetic progressions in B. Then the number of monochromatic 5-APs in $BB \cdots BR$ is*

$$\sum_{i \in S} a_i c_i t^2 + O(t).$$

In particular, we have

$$m_5(\mathbb{Z}_n) \leqslant \sum_{i \in S} a_i \frac{c_i}{b^2} + o(1).$$

Proof of Theorem 1.4. Recall the 2-coloring B_{74} of $\mathbb{Z}_{74}$ as following:

$$\{1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0,$$
$$1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0\}.$$

Table 3
The values of c_i's for B_{74} and various r's.

Values	c_0	c_1	c_2	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{13}	c_{14}	c_{15}
Even $r \neq 0$	146	293	377	377	378	359	293	293	359	378	377	377	293	146
Odd $r \neq 37$	146	293	375	375	374	357	293	293	357	374	375	375	293	146
$r = 37$	146	144	144	144	144	144	144	144	144	144	144	144	144	146

Write $n = 74t + r$, where $0 \leqslant r \leqslant 73$. If $r = 0$, then $m_5(\mathbb{Z}_n) \leqslant m_5(\mathbb{Z}_{74}) = \frac{73}{2738}$; we are through in this case. Now we assume $r \neq 0$ and use the periodic construction $B_{74}B_{74}\cdots B_{74}R$, where R is any bit-string of length r.

The values c_i in Lemma 3.3 depend on i and r. These values are given in Table 3.

By Lemma 3.3, we have

$$m_5(\mathbb{Z}_n) \leqslant \sum_{i \in S} a_i \frac{c_i}{b^2} + o(1) = \begin{cases} 3629/65\,712 + o(1) & \text{for odd } r \neq 37, \\ 289/10\,952 + o(1) & r = 37, \\ 3647/65\,712 + o(1) & \text{for even } r \neq 0. \end{cases}$$

The proof of Theorem 1.4 is finished. □

4. Proofs of Theorem 1.3 for even *n* and Theorem 1.5

Recall the coloring

$$B_{22} = (1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0).$$

We observe that the first 11 coordinates and last 11 coordinates differ only by 1 bit. Let $B_{11} = (1, 1, 1, 0, 1, *, 0, 1, 0, 0, 0)$, where '$*$' could be either 0 or 1. B_{11} has the following property. (This is because B_{22} contains no non-degenerate monochromatic 4-APs.)

Property 2. *No matter which bit-value the '$*$' takes, B_{11} contains no non-degenerate monochromatic 4-APs of $\mathbb{Z}_{11}$.*

Lemma 4.1. *For any $t \geqslant 2$, we have*

$$m_4(\mathbb{Z}_{11t}) \leqslant \frac{10 + m_4(\mathbb{Z}_t)}{121}. \tag{24}$$

Proof. Let B_t be a 2-coloring/bit-string of $\mathbb{Z}_t$ which has exactly $m_4(\mathbb{Z}_t)t^2$ monochromatic 4-APs. First we consider the periodic construction

$$\underbrace{B_{11}B_{11}\cdots B_{11}}_{t}.$$

Each block B_{11} has exactly one '$*$'; there are t $*$'s in total. Finally, we replace these $*$'s by the values of B_t in the cyclic order. We denote the coloring by $B_{11} \ltimes B_t$. (For example, $B_{22} = B_{11} \ltimes (1, 0)$.)

Because of Property 2, a 4-AP of $B_{11} \ltimes B_t$ with parameter (a, d) is monochromatic only if $11|d$. The number of monochromatic 4-APs is exactly $10t^2 + m_4(\mathbb{Z}_t)t^2$. We have

$$m_4(\mathbb{Z}_{11t}) \leqslant \frac{10t^2 + m_4(\mathbb{Z}_t)t^2}{(11t)^2} = \frac{10 + m_4(\mathbb{Z}_t)}{121}.$$

The proof of this lemma is finished. □

A similar construction can be applied to 5-APs. Let

$$B_{37} = (1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1,$$
$$0, *, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0).$$

Table 4
The values of c_i's for $B = B_{11} \ltimes B_{20}$ and even $r = 2, 4, \dots, 218$ such that r is not divisible by 22.

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
4882	7563	8230	7563	7563	8230	7563	4882

Let B_t be a 2-coloring/bit-string of $\mathbb{Z}_t$ which has exactly $m_5(\mathbb{Z}_t)t^2$ monochromatic 4-APs. We can define $B_{37} \ltimes B_t$ similarly. For example, $B_{74} = B_{37} \ltimes (1, 0)$. Note that B_{74} contains no non-degenerate monochromatic 5-APs. We have the following property.

Property 3. *No matter which bit-value the '$*$' takes, B_{37} contains no non-degenerate monochromatic 5-APs of $\mathbb{Z}_{37}$.*

Using this property and the construction $B_{37} \ltimes B_t$, we have the following lemma and the proof will be omitted.

Lemma 4.2. *For any $t \geqslant 2$, we have*

$$m_5(\mathbb{Z}_{37t}) \leqslant \frac{36 + m_4(\mathbb{Z}_t)}{37^2}. \tag{25}$$

Proof of Theorem 1.5. Applying Lemma 4.1 recursively, we have

$$m_4(\mathbb{Z}_{11^s}) \leqslant \frac{10}{11^2} + \frac{10}{11^4} + \cdots + \frac{10}{11^{2s}} + \frac{1}{11^{2s}} m_4(\mathbb{Z}_1) = \frac{1}{12} + \frac{1}{12 \times 11^{2s-1}}.$$

Thus,

$$\varliminf_{n\to\infty} m_4(\mathbb{Z}_n) \leqslant \varliminf_{s\to\infty} m_4(\mathbb{Z}_{11^s}) \leqslant \lim_{s\to\infty}\left(\frac{1}{12} + \frac{1}{12 \times 11^{2s-1}}\right) = \frac{1}{12}.$$

Similarly, from Lemma 4.2, we can show $\varliminf_{n\to\infty} m_5(\mathbb{Z}_n) \leqslant \frac{1}{38}$. □

Proof of Theorem 1.3 for even *n*. Here we assume n is even and not divisible by 22. Let $B = B_{11} \ltimes B_{20}$ which is a 2-coloring of $\mathbb{Z}_{220}$. Write $n = 220t + r$ with $0 \leqslant r \leqslant 218$. Here r is even and not divisible by 22. Consider the periodic construction $BB \cdots BR$ as before. For these r, the values c_i depends only on i but not on r. These values are given in Table 4.

By Lemma 3.2, we have

$$m_4(\mathbb{Z}_n) \leqslant \sum_{i=0}^{7} \frac{a_i c_i}{b^2} + o(1) = \frac{8543}{72\,600} + o(1) < 0.11767722.$$

The theorem is proved. □

5. Proof of Theorem 1.2

In this section, we will deal with lower bound of $m_4(\mathbb{Z}_n)$.

Proof of Theorem 1.2. Given a 2-coloring c of $\mathbb{Z}_n$, we will establish an inequality which is similar to Eq. (4.8) in [2]. For each $0 \leqslant i \leqslant 4$, let u_i be the number of 4-APs with exactly i red numbers. We have

$$\begin{aligned} u_1 + u_3 = {} & |A_1 \cap B_2 \cap B_3 \cap B_4| + |B_1 \cap A_2 \cap B_3 \cap B_4| + |B_1 \cap B_2 \cap A_3 \cap B_4| \\ & + |B_1 \cap B_2 \cap B_3 \cap A_4| + |B_1 \cap A_2 \cap A_3 \cap A_4| + |A_1 \cap B_2 \cap A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap B_3 \cap A_4| + |A_1 \cap A_2 \cap A_3 \cap B_4|. \end{aligned}$$

Note that

$$|A_1 \cap B_2 \cap B_3 \cap B_4| = |B_2 \cap B_3 \cap B_4| - |B_1 \cap B_2 \cap B_3 \cap B_4|. \tag{26}$$

Applying equations similar to (26), we get

$$4u_0 + u_1 + u_3 + 4u_4 = \sum_{1 \leqslant i < j < k \leqslant 4} \big(|A_i \cap A_j \cap A_k| + |B_i \cap B_j \cap B_k|\big). \tag{27}$$

By the inclusion–exclusion formula, for any $1 \leqslant i < j < k \leqslant 4$, we have

$$|A_i \cup A_j \cup A_k| = \sum_{s \in \{i,j,k\}} |A_s| - \sum_{\{s,t\} \in \binom{\{i,j,k\}}{2}} |A_s \cap A_t| + |A_i \cap A_j \cap A_k|.$$

Since $|A_i \cup A_j \cup A_k| = n^2 - |B_i \cap B_j \cap B_k|$, we have

$$|A_i \cap A_j \cap A_k| + |B_i \cap B_j \cap B_k| = n^2 - \sum_{s \in \{i,j,k\}} |A_s| + \sum_{\{s,t\} \in \binom{\{i,j,k\}}{2}} |A_s \cap A_t|. \tag{28}$$

By the symmetry of A_i's and B_i's, we get

$$|A_i \cap A_j \cap A_k| + |B_i \cap B_j \cap B_k| = n^2 - \sum_{s \in \{i,j,k\}} |B_s| + \sum_{\{s,t\} \in \binom{\{i,j,k\}}{2}} |B_s \cap B_t|. \tag{29}$$

Combining Eqs. (28) and (29) and summing over $1 \leqslant i < j < k \leqslant 4$, we get

$$\begin{aligned}
&2 \sum_{1 \leqslant i < j < k \leqslant 4} \big(|A_i \cap A_j \cap A_k| + |B_i \cap B_j \cap B_k|\big) \\
&\quad = \sum_{1 \leqslant i < j < k \leqslant 4} \left(2n^2 - \sum_{s \in \{i,j,k\}} \big(|A_s| + |B_s|\big) + \sum_{\{s,t\} \in \binom{\{i,j,k\}}{2}} \big(|A_s \cap A_t| + |B_s \cap B_t|\big)\right) \\
&\quad = 8n^2 - 12n^2 + 2 \sum_{1 \leqslant i < j \leqslant 4} \big(|A_i \cap A_j| + |B_i \cap B_j|\big) \\
&\quad = -4n^2 + 2 \sum_{1 \leqslant i < j \leqslant 4} \big(|A_i \cap A_j| + |B_i \cap B_j|\big).
\end{aligned}$$

Combining the equation above with Eq. (27), we have

$$4u_0 + u_1 + u_3 + 4u_4 = -2n^2 + \sum_{1 \leqslant i < j \leqslant 4} \big(|A_i \cap A_j| + |B_i \cap B_j|\big). \tag{30}$$

Lemma 2.1 implies $|A_i \cap A_j| \geqslant (\alpha n)^2$ and $|B_i \cap B_j| \geqslant (n - \alpha n)^2$ for $(i, j) \in \{(1, 2), (2, 3), (3, 4), (1, 4)\}$. We get

$$\begin{aligned}
u_1 + u_3 + 4u_0 + 4u_4 \geqslant{}& 2n^2 - 8\alpha n^2 + 8\alpha^2 n^2 + |A_1 \cap A_3| \\
&+ |A_2 \cap A_4| + |B_1 \cap B_3| + |B_2 \cap B_4|.
\end{aligned} \tag{31}$$

Let E be the collection of all even-colored 4-term progressions and O be the collection of all odd-colored 4-term progressions. We have $|E| = u_0 + u_2 + u_4$ and $|O| = u_1 + u_3$. Inequality (31) together with $\sum_{i=0}^{4} u_i = n^2$ give that

$$\begin{aligned} m_4(\mathbb{Z}_n, c) &= u_0 + u_4 \\ &= \frac{1}{4}\big(u_1 + u_3 + 4u_0 + 4u_4 + |E| - n^2\big) \\ &\geqslant \left(\frac{1}{4} - 2\alpha + 2\alpha^2\right)n^2 + \frac{|E|}{4} + \frac{1}{4}\big(|A_1 \cap A_3| + |B_1 \cap B_3|\big) \\ &\quad + \frac{1}{4}\big(|A_2 \cap A_4| + |B_2 \cap B_4|\big). \end{aligned} \tag{32}$$

We aim to modify the method in [11] to find a lower bound on $|E|$ which gives a lower bound on $m_4(\mathbb{Z}_n, c)$. Assume S is a 3-term progression in $\mathbb{Z}_n$. Let p_S be the number of even-colored 4-APs containing S and q_S be the number of odd-colored 4-APs containing S. Observe that $p_S + q_S = 2$. If $(a, a+d, a+2d)$ is a 3-AP, then it determines a pair of integers $x, y \in \mathbb{Z}_n$ such that $(x, a, a+d, a+2d)$ and $(a, a+d, a+2d, y)$ are two 4-APs containing S; the pair (x, y) is the *frame pair* of S. We have

$$\mathbb{E}_S p_S = 2|E| \quad \text{and} \quad \mathbb{E}_S q_S = 2|O|,$$

where the expectation operator $\mathbb{E}_S$ runs over all 3-APs. The following equality which ensures us to obtain a lower bound on E. We have

$$2|E| = 2|O| + \mathbb{E}_S(p_S - q_S) = 2\big(n^2 - |E|\big) - \mathbb{E}_S\big(|p_S - q_S|\big) + 2\mathbb{E}_S(p_S - q_S | p_S > q_S). \tag{33}$$

Solving for $|E|$ in Eq. (33) gives

$$|E| = \frac{1}{2}n^2 - \frac{1}{4}\mathbb{E}_S\big(|p_S - q_S|\big) + \frac{1}{2}\mathbb{E}_S(p_S - q_S | p_S > q_S). \tag{34}$$

Eq. (34) was first proved by Wolf [11] for $\mathbb{Z}_p$, where the last term $\mathbb{E}_S(p_S - q_S | p_S > q_S)$ was thrown away. Note that $\mathbb{E}_S(p_S - q_S | p_S > q_S)$ is the number of monochromatic 5-APs with some patterns and it is a non-trivial term. We have the following claim, whose proof will be postponed until the end of the section.

Claim 5.1. $\mathbb{E}_S(p_S - q_S | p_S > q_S) \geqslant n^2/12$ *for any positive integer n.*

Observe that $|p_S - q_S| \neq 0$ if and only if the frame pair of S is monochromatic. Furthermore, $|p_S - q_S| = 2$ if $|p_S - q_S| \neq 0$. Note that when n is prime, each frame pair belongs to a unique 3-term progression as 4 is invertible in $\mathbb{Z}_n$. However, if n is not prime, then each frame pair may belong to more than one 3-term progression or not belong to any 3-term progressions. We will compute the value of $E_S(|p_S - q_S|)$ case by case according to n modulo 4.

Case 1: $n \equiv 1, 3 \mod 4$. In this case, each frame pair belongs to a unique 3-term progression since 4 is invertible in $\mathbb{Z}_n$. The term $\mathbb{E}_S(|p_S - q_s|)$ is twice the number of monochromatic pairs in the coloring c, that is $\mathbb{E}_S(|p_S - q_s|) = 2(\alpha n)^2 + 2(n - \alpha n)^2$. We obtain

$$|E| \geqslant \alpha(1-\alpha)n^2 + \frac{n^2}{24}.$$

By Lemma 2.1, we have $|A_1 \cap A_3| = |A_2 \cap A_4| \geqslant (\alpha n)^2$ and $|B_1 \cap B_3| = |B_2 \cap B_4| \geqslant (n - \alpha n)^2$. Therefore, in this case, inequality (32) is

$$m_4(\mathbb{Z}_n, c) \geqslant \frac{(3 - 11\alpha - 11\alpha^2)n^2}{4} + \frac{n^2}{96}. \tag{35}$$

It is straightforward to check that the minimum value of the right-hand side of inequality (35) is $7n^2/96$ and it is achieved at $\alpha = 1/2$. We have $m_4(\mathbb{Z}_n) \geqslant 7/96$ in this case.

Case 2: $n \equiv 2 \mod 4$. For $0 \leqslant i \leqslant 3$, let $\mathbb{Z}_n^i = \{z \in \mathbb{Z}_n\colon\ z \equiv i \mod 4\}$. Let $a_i = |A \cap \mathbb{Z}_n^i|$ and $b_i = |B \cap \mathbb{Z}_n^i|$.

A pair (x, y) is a frame pair if and only if $y - x = 4d$ for some $d \in \mathbb{Z}_n$. Assume $n = 4r + 2$. If d is a solution for $4d = y - x$, then $d + 2r + 1$ is another solution. We have

$$\mathbb{E}_S\big(|p_S - q_S|\big) = 4(a_0+a_2)^2 + 4(a_1+a_3)^2 + 4(b_0+b_2)^2 + 4(b_1+b_3)^2. \tag{36}$$

By the same argument, we have

$$|A_1 \cap A_3| = |A_2 \cap A_4| = 2(a_0+a_2)^2 + 2(a_1+a_3)^2$$

and

$$|B_1 \cap B_3| = |B_2 \cap B_4| = 2(b_0+b_2)^2 + 2(b_1+b_3)^2.$$

Therefore, $|E| + |A_1 \cap A_3| + |A_2 \cap A_4| + |B_1 \cap B_3| + |B_2 \cap B_4|$ is at least

$$\frac{1}{2}n^2 + 3\big((a_0+a_2)^2 + (a_1+a_3)^2 + (b_0+b_2)^2 + (b_1+b_3)^2\big) + \frac{n^2}{24}.$$

We have the following inequality

$$\begin{aligned}
&|E| + |A_1 \cap A_3| + |A_2 \cap A_4| + |B_1 \cap B_3| + |B_2 \cap B_4| \\
&\quad \geqslant \frac{1}{2}n^2 + \frac{3}{2}\left(\sum_{i=0}^{3} a_i\right)^2 + \frac{3}{2}\left(\sum_{i=0}^{3} b_i\right)^2 + \frac{n^2}{24} \\
&\quad = \left(\frac{1}{2} + \frac{3}{2}\alpha^2 + \frac{3}{2}(1-\alpha)^2 + \frac{1}{24}\right)n^2.
\end{aligned} \tag{37}$$

Combining inequalities (32) and (37), we get

$$m_4(\mathbb{Z}_n, c) \geqslant \frac{(3 - 11\alpha + 11\alpha^2)n^2}{4} + \frac{n^2}{96}.$$

Note the minimum is reached at $\alpha = 1/2$. It follows $m_4(\mathbb{Z}_n) \geqslant 7/96$.

Case 3: $n \equiv 0 \mod 4$. The method above fails in this case, which suggests that it is possible to find a 2-coloring of $\mathbb{Z}_n$ which contains few monochromatic 4-term progressions. Replacing the terms on the right-hand side of inequality (31) by the lower bounds from Lemma 2.1, we obtain

$$u_1 + u_3 + 4u_0 + 4u_4 \geqslant 4n^2 - 12\alpha n^2 + 12\alpha^2 n^2. \tag{38}$$

Combining with $\sum_{i=0}^{4} u_i = n^2$, we have

$$u_0 + u_4 \geqslant \frac{u_2}{3} + 1 - 4\alpha n^2 + 4\alpha^2 n^2 \geqslant \frac{u_2}{3}. \tag{39}$$

The remark following the proof of Theorem 4.4 in [2] gives

$$u_0 + u_2 + u_4 \geqslant \frac{8n^2}{33}. \tag{40}$$

Combining inequalities (39) and (40), we get

$$m_4(\mathbb{Z}_n, c) = u_0 + u_4 \geqslant \frac{2n^2}{33}.$$

This implies $m_4(\mathbb{Z}_n) \geqslant 2/33$. We completed the proof of Theorem 1.2. □

We finish this section by proving Claim 5.1.

Proof of Claim 5.1. Observe that $p_S > q_S$ if and only if the coloring pattern of 5-APs (S and its frame pair (x, y)) is in the following set:

$$\begin{aligned}
F = \{&(1,1,1,1,1), (1,0,0,1,1), (1,0,1,0,1), (1,1,0,0,1), \\
&(0,0,0,0,0), (0,1,1,0,0), (0,1,0,1,0), (0,0,1,1,0)\}.
\end{aligned}$$

Moreover, for each S, $p_S - q_S = 2$ if $p_S > q_S$. Therefore the value of $\mathbb{E}_S(p_S - q_S | p_S > q_S)$ is twice the number of increasing 5-term progressions with coloring patterns from F. Using an exhaustive search, one can show that for any 2-coloring of [46], there is at least one increasing 5-AP with coloring pattern in F.

A further computation shows that any 2-coloring of [74] contain at least 27 increasing 5-APs with coloring patterns in F. (We use the breadth-first search algorithm running on the binary trees of depth 74. The program was implemented in C++ using a GPU. It took several hours to complete the search.) Note that the number of increasing 5-APs in [74] with $d = 1$ is 70, the number of 5-APs in [74] with $d = 2$ is 66, etc. The number of 5-APs in [74] is

$$70 + 66 + 62 + \cdots + 6 + 2 = 648.$$

For any 2-coloring of $\mathbb{Z}_n$, the number of 74-APs is exactly n^2; each of them (degenerate or not) contains 27 5-APs with coloring patterns in F. Each 5-AP with coloring pattern in F is counted at most 648-times. Thus, the number of 5-APs with coloring pattern in F is at least

$$\frac{27}{648}n^2 = \frac{1}{24}n^2.$$

Thus we have

$$\mathbb{E}_S(p_S - q_S | p_S > q_S) \geqslant \frac{n^2}{12}.$$

We finished the proof of the claim. $\square$

Appendix A

Proof of Lemma 3.1. Since P has m vertices, let $v_1, \ldots, v_m$ be the vertices of the polygon $v + tP$. For $i = 1, \ldots, m$, let u_i be a grid point closest to v_i. (If there is more than one choice, then break ties arbitrarily.) We have $|u_i v_i| \leqslant \frac{\sqrt{2}}{2}$. Let Q be the polygon with vertices $u_1, u_2, \ldots, u_m$. (For convenience, we write $v_{m+1} = v_0$ and $u_{m+1} = u_0$.) The polygon Q can be viewed as an approximation of the polygon $v + tP$; thus Q is simple for sufficiently large t.

Applying Pick's theorem to Q, we have

$$A(Q) - I(Q) = \frac{B(Q)}{2} - 1.$$

We observe that the number of grid points on any line segment of length l is at most $l + 1$. We have

$$\begin{aligned}
B(Q) &\leqslant \sum_{i=1}^{m} \bigl(|u_i u_{i+1}| + 1\bigr) \\
&\leqslant \sum_{i=1}^{m} \bigl(|v_i v_{i+1}| + |u_i v_i| + |u_{i+1} v_{i+1}| + 1\bigr) \\
&\leqslant \sum_{i=1}^{m} \bigl(|v_i v_{i+1}| + \sqrt{2} + 1\bigr) \\
&= tL + (\sqrt{2} + 1)m.
\end{aligned}$$

Let S_i be the convex region spanned by v_i, v_{i+1}, u_i, u_{i+1}. Note S_i is covered by four triangles $\Delta u_i v_i v_{i+1}$, $\Delta u_i v_i u_{i+1}$, $\Delta u_{i+1} v_{i+1} u_i$, and $\Delta u_{i+1} v_{i+1} v_i$ exactly twice. We have

$$\begin{aligned}
A(S_i) &= \frac{1}{2}\bigl(A(\Delta u_i v_i v_{i+1}) + A(\Delta u_i v_i u_{i+1}) + A(\Delta u_{i+1} v_{i+1} u_i) + A(\Delta u_{i+1} v_{i+1} v_i)\bigr) \\
&\leqslant \frac{1}{2}\bigl(|v_i v_{i+1}| + |u_i u_{i+1}|\bigr)\frac{\sqrt{2}}{2}
\end{aligned}$$

$$\leqslant \frac{\sqrt{2}}{4}\big(|v_iv_{i+1}| + |u_iv_i| + |v_iv_{i+1}| + |v_{i+1}u_{i+1}|\big)$$
$$\leqslant \frac{\sqrt{2}}{4}\big(2|v_iv_{i+1}| + \sqrt{2}\big)$$
$$= \frac{\sqrt{2}}{2}|v_iv_{i+1}| + \frac{1}{2}.$$

Summing up, we get

$$\big|A(Q) - A(v+tP)\big| \leqslant \sum_{i=1}^{m} A(S_i) \leqslant \sum_{i=1}^{m}\left(\frac{\sqrt{2}}{2}|v_iv_{i+1}| + \frac{1}{2}\right) = \frac{\sqrt{2}}{2}Lt + \frac{m}{2}.$$

Let T_i be the set of grid points inside S_i or on the line segment u_iu_{i+1}. Let P_i be the convex set spanned by T_i. Applying Pick's theorem to P_i, we have

$$A(P_i) = I(P_i) + \frac{B(P_i)}{2} - 1.$$

Thus

$$|T_i| = I(P_i) + B(P_i) \leqslant 2\big(A(P_i) + 1\big) \leqslant 2\big(A(S_i) + 1\big) \leqslant \sqrt{2}|v_iv_{i+1}| + 3.$$

Summing up, we get

$$\big|I(Q) - I(v+tP)\big| \leqslant \sum_{i=1}^{m} |T_i| \leqslant \sum_{i=1}^{m} \sqrt{2}|v_iv_{i+1}| + 3 = \sqrt{2}tL + 3m.$$

Putting together, we have

$$\begin{aligned}\big|I(v+tP) - A(v+tP)\big| &\leqslant \big|I(Q) - A(Q)\big| + \big|A(Q) - A(v+tP)\big| + \big|I(Q) - I(v+tp)\big| \\ &\leqslant \frac{1}{2}\big(tL + (\sqrt{2}+1)m\big) - 1 + \left(\frac{\sqrt{2}}{2}tL + \frac{m}{2}\right) + (\sqrt{2}tL + 3m) \\ &= \frac{3\sqrt{2}+1}{2}Lt + \left(4 + \frac{\sqrt{2}}{2}\right)m - 1 \\ &< 3Lt + 5m.\end{aligned}$$

The proof of this lemma is finished. $\square$

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